Duration : 144 minutes

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Linear Algebra Exam

Common part

Fall 2018

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2018.

Notation

– For a matrix A, a_{ij} denotes the entry of A in row i and column j.
– For a vector $\boldsymbol{x}$, x_i denotes the i-the coordinate of $\boldsymbol{x}$.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} -3 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \end{pmatrix} \right\}$$

be two bases of $\mathbb{R}^2$. Let $P_{\mathcal{B}\to\mathcal{C}}$ be the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$; in other words, we have $\left[\boldsymbol{x}\right]_{\mathcal{C}} = P_{\mathcal{B}\to\mathcal{C}}\left[\boldsymbol{x}\right]_{\mathcal{B}}$ for all $\boldsymbol{x} \in \mathbb{R}^2$. Then

☐ $P_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} 11 & 7 \\ -30 & -19 \end{pmatrix}$

☐ $P_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} -19 & -7 \\ 30 & 11 \end{pmatrix}$

☐ $P_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} -9 & 5 \\ -2 & 1 \end{pmatrix}$

■ $P_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} 1 & -5 \\ 2 & -9 \end{pmatrix}$

Question 2 : Let

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad W = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Then the orthogonal projection of $\boldsymbol{v}$ onto W is

■ $\begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}$

☐ $\begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix}$

☐ $\begin{pmatrix} 1 \\ 1/2 \\ -1/2 \end{pmatrix}$

☐ $\begin{pmatrix} -2/3 \\ 2/3 \\ -2/3 \end{pmatrix}$

Question 3 : Let

$$A = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1/6 & 0 & 1/3 & 1/3 \end{pmatrix}$$

and let B be a $4{\times}4$ matrix such that $AB = I_4$.
The trace of B is defined as $\mathrm{Tr}(B) = b_{11} + b_{22} + b_{33} + b_{44}$. Then

- ☐ $\mathrm{Tr}(B) = 3$
- ☐ $\mathrm{Tr}(B) = 2$
- ☐ $\mathrm{Tr}(B) = 1$
- ■ $\mathrm{Tr}(B) = 5$

Question 4 : Let

$$\boldsymbol{x}_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{x}_3 = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 7 \end{pmatrix}$$

and let $W = \mathrm{Span}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$. Applying Gram-Schmidt orthogonalization to the basis $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ of W, without normalizing or changing the order, yields an orthogonal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ of W, where

- ☐ $\boldsymbol{v}_3 = \boldsymbol{x}_3 - \boldsymbol{v}_1 + \boldsymbol{v}_2$
- ☐ $\boldsymbol{v}_3 = \boldsymbol{x}_3 + 9\boldsymbol{v}_1 - 9\boldsymbol{v}_2$
- ■ $\boldsymbol{v}_3 = \boldsymbol{x}_3 + \boldsymbol{v}_1 - \boldsymbol{v}_2$
- ☐ $\boldsymbol{v}_3 = \boldsymbol{x}_3$

Question 5 : Let α be a real number and let

$$A = \begin{pmatrix} -3 & -3 & -3 & -2 \\ -2 & -1 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ \alpha & 2 & 3 & 3 \end{pmatrix}.$$

The determinant of A is

- ☐ $\det(A) = -2$
- ☐ $\det(A) = 2 - \alpha$
- ☐ $\det(A) = -2 - \alpha$
- ■ $\det(A) = 2$

Question 6 : Let A be an $m \times n$ matrix such that $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for every $\boldsymbol{b} \in \mathbb{R}^m$. Which one of the following statements is always true?

☐ $\dim\big(\mathrm{Col}(A^T)\big) = n$

■ $A^T\boldsymbol{y} = \boldsymbol{0}$ has a unique solution

☐ $\dim\big(\mathrm{Nul}(A)\big) = 0$

☐ $A^T\boldsymbol{y} = \boldsymbol{c}$ has at least one solution for every $\boldsymbol{c} \in \mathbb{R}^n$

Question 7 : Let

$$A = \begin{pmatrix} 2 & 1 & -1 & 1 \\ 0 & 3 & 2 & -1 \\ 1 & -1 & 4 & -1 \\ -1 & 1 & 1 & 3 \end{pmatrix}$$

and $V = \big\{\boldsymbol{x} \in \mathbb{R}^4 \mid A\boldsymbol{x} = 3\boldsymbol{x}\big\}$. Then

■ $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}\right\}$

☐ $V = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

☐ $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ -1/2 \\ 1 \end{pmatrix}\right\}$

☐ $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1/2 \end{pmatrix}\right\}$

Question 8 : Let A be an $m \times n$ matrix. If $m < n$, then the smallest possible value of $\dim(\mathrm{Nul}\,A)$ is

☐ $m - n$

☐ 0

■ $n - m$

☐ m

Question 9 : Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by
$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} 6x_1 \\ -12x_1 + 6x_2 - 3x_3 + 6x_4 \\ -24x_1 + 18x_3 \\ -12x_1 + 6x_2 + 6x_3 + 6x_4 \end{pmatrix}.$$
Then

☐ $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 3 \\ 0 \end{pmatrix}\right\}$

☐ $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -2 \\ -4 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

■ $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -2 \\ -4 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -6 \\ -2 \end{pmatrix}\right\}$

☐ $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

Question 10 : Let
$$A = \begin{pmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{pmatrix}.$$
Then the eigenvalues of A are

☐ -5, -1, and 2

☐ -5, -2, and 3

■ -2, 1, and 5

☐ -3, 2, and 5

Question 11 : Let $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_6\}$ be an orthonormal basis of $\mathbb{R}^6$, and let
$$A = 3\boldsymbol{v}_1\boldsymbol{v}_1^T - 2\left(\boldsymbol{v}_2\boldsymbol{v}_2^T + \boldsymbol{v}_3\boldsymbol{v}_3^T\right) + \tfrac{1}{3}\left(\boldsymbol{v}_4\boldsymbol{v}_4^T + \boldsymbol{v}_5\boldsymbol{v}_5^T + \boldsymbol{v}_6\boldsymbol{v}_6^T\right).$$
Then the characteristic polynomial p_A of A is

☐ $p_A(t) = (t-3) + 2(t+2) + 3(t-\frac{1}{3})$

■ $p_A(t) = (t-3)(t+2)^2(t-\frac{1}{3})^3$

☐ $p_A(t) = (t-3) + (t+2)^2 + (t-\frac{1}{3})^3$

☐ $p_A(t) = t^3(t-3)(t+2)(t-\frac{1}{3})$

Question 12 : Let $\mathcal{B} = \{1, 1+t, 1+t^2\}$ be a basis of $\mathbb{P}_2$, and let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(a+bt+ct^2) = (a+b+c) + (a-b)t + (b-c)t^2 \qquad \text{for all } a,b,c \in \mathbb{R}.$$

Let $M = [T]_{\mathcal{B}\to\mathcal{B}}$ be the matrix that represents T in the basis $\mathcal{B}$; in other words, M is the matrix such that $\big[T(p)\big]_{\mathcal{B}} = M\,\big[p\big]_{\mathcal{B}}$ for all $p \in \mathbb{P}_2$. Then

☐ $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

☐ $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$

☐ $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

■ $M = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Question 13 : Let h be a real parameter, and let

$$A_1 = \begin{pmatrix} 1 & -1 \\ -1 & -h \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ h & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & h \\ 1 & 0 \end{pmatrix}.$$

Then the matrices A_1, A_2, A_3 and A_4 are linearly dependent if and only if

☐ $h = -1$, $h = 0$, or $h = 1$

■ $h = 0$ or $h = 1$

☐ $h = 0$

☐ $h = -1$ or $h = 0$

Question 14 : Let h be a real parameter, and let

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & \frac{7}{3} & 1 \\ -3 & 1-2h & 1-h \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ \frac{4}{3}h + \frac{2}{3} \\ -1 \end{pmatrix}.$$

Then the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$

☐ has the solution $\boldsymbol{x} = \begin{pmatrix} \frac{1}{6}(4+h+h^2+1) \\ \frac{1}{2}(h-h^2-1) \\ h^2+1 \end{pmatrix}$ for any choice of h

■ has no solution for any choice of h

☐ has the solution $\boldsymbol{x} = \begin{pmatrix} \frac{1}{6}(4+h) \\ \frac{1}{2}h \\ 0 \end{pmatrix}$ if and only if $h \neq 1$ and $h \neq -1$

☐ has the solution $\boldsymbol{x} = \begin{pmatrix} \frac{1}{6}(4+h) \\ \frac{1}{2}h \\ 0 \end{pmatrix}$ if and only if $h = 1$ or $h = -1$

Question 15 : Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} \quad \text{be a basis of } \mathbb{R}^2, \quad \text{and let} \quad M = \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix}$$

be the matrix $M = [T]_{\mathcal{B} \to \mathcal{B}}$ that represents T in the basis $\mathcal{B}$; in other words, M is the matrix such that $\big[T(\boldsymbol{x})\big]_{\mathcal{B}} = M\big[\boldsymbol{x}\big]_{\mathcal{B}}$ for all $\boldsymbol{x} \in \mathbb{R}^2$. Then

■ $T(\boldsymbol{x}) = \begin{pmatrix} -x_1 - 2x_2 \\ 9x_1 + 9x_2 \end{pmatrix}$

□ $T(\boldsymbol{x}) = \begin{pmatrix} -7x_1 + 19x_2 \\ -6x_1 + 15x_2 \end{pmatrix}$

□ $T(\boldsymbol{x}) = \dfrac{1}{18}\begin{pmatrix} 4x_1 - 7x_2 \\ -9x_1 + 18x_2 \end{pmatrix}$

□ $T(\boldsymbol{x}) = \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + 6x_2 \end{pmatrix}$

Question 16 : Let $\mathbf{M}_{2\times3}$ be the vector space of 2×3 matrices.
Which of the following three subsets of $\mathbf{M}_{2\times3}$ are subspaces of $\mathbf{M}_{2\times3}$?

$$\mathcal{E}_1 = \left\{ \begin{pmatrix} u & 0 & v \\ 0 & w & 0 \end{pmatrix} \,\middle|\, u, v, w \in \mathbb{R} \text{ and } uv = w^2 \right\},$$

$$\mathcal{E}_2 = \left\{ a\begin{pmatrix} 1 & 3/2 & 7 \\ -5 & \sqrt{2} & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\},$$

$$\mathcal{E}_3 = \left\{ \begin{pmatrix} 0 & x & 1 \\ y & 0 & x-y \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

■ only $\mathcal{E}_2$

□ only $\mathcal{E}_3$

□ only $\mathcal{E}_1$

□ only $\mathcal{E}_2$ and $\mathcal{E}_3$

Question 17 : Let $T : \mathbb{P}_3 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(a + bt + ct^2 + dt^3\right) = \begin{pmatrix} a + b - c + 3d \\ b + 2d \\ 2a + 3b - 2c + 8d \\ -3b - 6d \end{pmatrix}.$$

Then

□ $\operatorname{Ker} T = \operatorname{Span}\left\{4 + 2t + 3t^2 - t^3 \,,\ t + 2t^3\right\}$

■ $\operatorname{Ker} T = \operatorname{Span}\left\{1 + t^2 \,,\ 1 + 2t - t^3\right\}$

□ $\operatorname{Ker} T = \operatorname{Span}\left\{2t - t^2 - t^3 \,,\ 2 + t^2\right\}$

□ $\operatorname{Ker} T = \operatorname{Span}\left\{t + t^3 \,,\ 1 + 2t^2 - t^3\right\}$

Question 18 : The matrix $A = \begin{pmatrix} 23 & -36 \\ -36 & 2 \end{pmatrix}$ is orthogonally diagonalizable, i.e., it can be written as $A = QDQ^T$, for an orthogonal matrix Q and a diagonal matrix D, where

☐ $Q = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

■ $Q = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$ and $D = \begin{pmatrix} -25 & 0 \\ 0 & 50 \end{pmatrix}$

☐ $Q = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

☐ $Q = \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}$ and $D = \begin{pmatrix} -25 & 0 \\ 0 & 50 \end{pmatrix}$

Question 19 : Let A be an $m \times n$ matrix whose entries are not all 0, and let $\boldsymbol{b} \in \mathbb{R}^m$. Which one of the following statements is always true?

☐ There is a unique vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{b} - A\boldsymbol{x}$ belongs to $\mathrm{Nul}(A^T)$

☐ The matrix A^TA is invertible

☐ The equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique least-squares solution

■ $A\widehat{\boldsymbol{x}} = A\widehat{\boldsymbol{x}}'$ if $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{x}}'$ are two least-squares solutions of $A\boldsymbol{x} = \boldsymbol{b}$

Question 20 : Let

$$A = \begin{pmatrix} 1 & -2 & -3 & 4 \\ 5 & -4 & -17 & 16 \\ -2 & 16 & -3 & -11 \\ 3 & -15 & -2 & 6 \end{pmatrix}.$$

Compute the LU factorization of A (using only the elementary row operation that adds a multiple of one row to another row below it). Then the entry ℓ_{42} of the matrix L equals

☐ $\ell_{42} = -\dfrac{2}{3}$

☐ $\ell_{42} = \dfrac{3}{2}$

☐ $\ell_{42} = \dfrac{2}{3}$

■ $\ell_{42} = -\dfrac{3}{2}$

Question 21 : Let A and B be invertible $n \times n$ matrices. Then the number

$$\frac{\det(A^T) + \det(B^T)}{\det(A)\det(B)}$$

- ☐ is equal to $\det(A^T - A) + \det(B^T - B)$
- ■ is equal to $\det(A^{-1}) + \det(B^{-1})$
- ☐ is equal to $\det(B^{-1} + A^{-1})$
- ☐ is equal to $\dfrac{1}{\det(B)} - \dfrac{1}{\det(A)}$

Question 22 : Let A and B be diagonalizable 3×3 matrices.
Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis of $\mathbb{R}^3$ such that

(a) the eigenspaces of A are $E_1 = \text{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $E_2 = \text{Span}\{\boldsymbol{u}_3\}$, and

(b) $B\boldsymbol{u}_2 = -\boldsymbol{u}_2$ and $\text{Nul}(B) = \text{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_3\}$.

Then

- ☐ none of the matrices AB and $A+B$ are always diagonalizable
- ☐ the matrix $A+B$ is always diagonalizable, but AB is not always diagonalizable
- ☐ the matrix AB is always diagonalizable, but $A+B$ is not always diagonalizable
- ■ the matrices AB and $A+B$ are both always diagonalizable

Question 23 : Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 4 \end{pmatrix}.$$

Then the least-squares solution $\widehat{\boldsymbol{x}} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{pmatrix}$ of the equation $A\boldsymbol{x} = \boldsymbol{b}$ satisfies

- ☐ $\widehat{x}_1 = 4/5$ and $\widehat{x}_2 = 1$
- ■ $\widehat{x}_1 = 8/5$ and $\widehat{x}_3 = 0$
- ☐ $\widehat{x}_1 = 4/5$ and $\widehat{x}_2 = 0$
- ☐ $\widehat{x}_1 = 8/5$ and $\widehat{x}_3 = 1$

Question 24 : Let b be a real parameter, and let

$$A = \begin{pmatrix} 1 & b-1 & 0 \\ 0 & b & 0 \\ 0 & b+1 & b \end{pmatrix}.$$

Which one of the following statements is true?

- ■ If $b = -1$, then A has two distinct eigenvalues, and A is diagonalizable
- ☐ If $b \neq 1$ and $b \neq -1$, then A has two distinct eigenvalues, and A is diagonalizable
- ☐ If $b = 1$, then A has one eigenvalue, and A is diagonalizable
- ☐ If $b \neq 1$ and $b \neq -1$, then A has one eigenvalue, and A is diagonalizable